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CANONICAL BASES AND AFFINE HECKE ALGEBRAS OF TYPE D

P. SHAN, M. VARAGNOLO, E. VASSEROT

ABSTRACT. We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. The proof is similar to the proof of the type B case in [VV].

INTRODUCTION

Let \mathbf{f} be the negative part of the quantized enveloping algebra of type $A^{(1)}$. Lusztig's description of the canonical basis of \mathbf{f} implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [G], and was used in [A]. More precisely, there is a linear isomorphism between \mathbf{f} and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on \mathbf{f} .

The branching rules for affine Hecke algebras of type B have been investigated quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an analogue of Ariki's construction is conjectured and studied for affine Hecke algebras of type B. Here \mathbf{f} is replaced by a module ${}^\theta\mathbf{V}(\lambda)$ over an algebra ${}^\theta\mathbf{B}$. More precisely it is conjectured there that ${}^\theta\mathbf{V}(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B should be given by the ${}^\theta\mathbf{B}$ -action on ${}^\theta\mathbf{V}(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type D has also been conjectured in [KM]. In this case \mathbf{f} is replaced by another module ${}^\circ\mathbf{V}$ over the algebra ${}^\theta\mathbf{B}$ (the same algebra as in the type B case). The purpose of this paper is to prove the type D case. The method of the proof is the same as in [VV]. First we introduce a family of graded algebras ${}^\circ\mathbf{R}_m$ for m a non negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This complex enters in the Kazhdan-Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify ${}^\circ\mathbf{V}$ with the sum of the Grothendieck groups of the graded algebras ${}^\circ\mathbf{R}_m$.

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The plan of the paper is the following. In Section 1 we introduce a graded algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$. It is associated with a quiver Γ with an involution θ and with a dimension vector ν . In Section 2 we consider a particular choice of pair (Γ, θ) . The graded algebras ${}^\circ\mathbf{R}(\Gamma)_\nu$ associated with this pair (Γ, θ) are denoted by the symbol ${}^\circ\mathbf{R}_m$. Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group $SO(2m)$, and we prove that it is Morita equivalent to ${}^\circ\mathbf{R}_m$. In Section 3 we categorify the module ${}^\circ\mathbf{V}$ from [KM] using the graded algebras ${}^\circ\mathbf{R}_m$, see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. NOTATION

0.1. Graded modules over graded algebras. Let \mathbf{k} be an algebraically closed field of characteristic 0. By a graded \mathbf{k} -algebra $\mathbf{R} = \bigoplus_d \mathbf{R}_d$ we'll always mean a \mathbb{Z} -graded associative \mathbf{k} -algebra. Let $\mathbf{R}\text{-mod}$ be the category of finitely generated graded \mathbf{R} -modules, $\mathbf{R}\text{-fmod}$ be the full subcategory of finite-dimensional graded modules and $\mathbf{R}\text{-proj}$ be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We'll abbreviate

$$K(\mathbf{R}) = [\mathbf{R}\text{-proj}], \quad G(\mathbf{R}) = [\mathbf{R}\text{-fmod}].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category \mathcal{C} . Assume that the \mathbf{k} -vector spaces \mathbf{R}_d are finite dimensional for each d . Then $K(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in $\mathbf{R}\text{-proj}$, and $G(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in $\mathbf{R}\text{-fmod}$. Given an object M of $\mathbf{R}\text{-proj}$ or $\mathbf{R}\text{-fmod}$ let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M = [M]$. We'll write $[M : N]$ for the composition multiplicity of the \mathbf{R} -module N in the \mathbf{R} -module M . Consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. If the grading of \mathbf{R} is bounded below then the \mathcal{A} -modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here \mathcal{A} acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM = M[1], \quad v^{-1}M = M[-1].$$

For any M, N in $\mathbf{R}\text{-mod}$ let

$$\text{hom}_{\mathbf{R}}(M, N) = \bigoplus_d \text{Hom}_{\mathbf{R}}(M, N[d])$$

be the \mathbb{Z} -graded \mathbf{k} -vector space of all \mathbf{R} -module homomorphisms. If $\mathbf{R} = \mathbf{k}$ we'll omit the subscript \mathbf{R} in hom 's and in tensor products. For any graded \mathbf{k} -vector space $M = \bigoplus_d M_d$ we'll write

$$\text{gdim}(M) = \sum_d v^d \dim(M_d),$$

where \dim is the dimension over \mathbf{k} .

0.2. Quivers with involutions. Recall that a quiver Γ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where I is the set of vertices, H is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of h respectively. Note that the set I may be infinite. We'll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write

$$H_{i,j} = \{h \in H; h' = i, h'' = j\}.$$

We'll abbreviate $i \rightarrow j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

An involution θ on Γ is a pair of involutions on I and H , both denoted by θ , such that the following properties hold for each h in H

- $\theta(h)' = \theta(h'')$ and $\theta(h)'' = \theta(h')$,
- $\theta(h') = h''$ iff $\theta(h) = h$.

We'll always assume that θ has no fixed points in I , i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that θ has no fixed point. Let

$${}^\theta \mathbb{N}I = \{\nu = \sum_i \nu_i i \in \mathbb{N}I : \nu_{\theta(i)} = \nu_i, \forall i\}.$$

For any $\nu \in {}^\theta \mathbb{N}I$ set $|\nu| = \sum_i \nu_i$. It is an even integer. Write $|\nu| = 2m$ with $m \in \mathbb{N}$. We'll denote by ${}^\theta I^\nu$ the set of sequences

$$\mathbf{i} = (i_{1-m}, \dots, i_{m-1}, i_m)$$

of elements in I such that $\theta(i_l) = i_{1-l}$ and $\sum_k i_k = \nu$. For any such sequence \mathbf{i} we'll abbreviate $\theta(\mathbf{i}) = (\theta(i_{1-m}), \dots, \theta(i_{m-1}), \theta(i_m))$. Finally, we set

$${}^\theta I^m = \bigcup_{\nu} {}^\theta I^\nu, \quad \nu \in {}^\theta \mathbb{N}I, \quad |\nu| = 2m.$$

0.3. The wreath product. Given a positive integer m , let \mathfrak{S}_m be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. Write s_1, \dots, s_{m-1} for the simple reflections in \mathfrak{S}_m . For each $l = 1, 2, \dots, m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l -th position. There is a unique action of W_m on the set $\{1-m, \dots, m-1, m\}$ such that \mathfrak{S}_m permutes $1, 2, \dots, m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^\theta I^\nu$. Indeed, view a sequence \mathbf{i} as the map

$$\{1-m, \dots, m-1, m\} \rightarrow I, \quad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$. For each ν we fix once for all a sequence

$$\mathbf{i}_e = (i_{1-m}, \dots, i_m) \in {}^\theta I^\nu.$$

Let W_e be the centralizer of \mathbf{i}_e in W_m . Then there is a bijection

$$W_e \backslash W_m \rightarrow {}^\theta I^\nu, \quad W_e w \mapsto w^{-1}(\mathbf{i}_e).$$

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let ${}^\circ W_m$ be the subgroup of W_m generated by s_0, \dots, s_{m-1} . We'll regard it as a Weyl group of type D_m such that s_0, \dots, s_{m-1} are the simple reflections. Note that W_e is a subgroup of ${}^\circ W_m$. Indeed, if $W_e \not\subset {}^\circ W_m$ there should exist l such that ε_l belongs to W_e . This would imply that $i_l = \theta(i_l)$, contradicting the fact that θ has no fixed point. Therefore ${}^\theta I^\nu$ decomposes into two ${}^\circ W_m$ -orbits. We'll denote them by ${}^\theta I_+^\nu$ and ${}^\theta I_-^\nu$. For $m = 1$ we set ${}^\circ W_1 = \{e\}$ and we choose again ${}^\theta I_+^\nu$ and ${}^\theta I_-^\nu$ in a obvious way.

1. THE GRADED \mathbf{k} -ALGEBRA ${}^\circ\mathbf{R}(\Gamma)_\nu$

Fix a quiver Γ with set of vertices I and set of arrows H . Fix an involution θ on Γ . Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbf{NI}$. Set $|\nu| = 2m$.

1.1. Definition of the graded \mathbf{k} -algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$. Assume that $m > 1$. We define a graded \mathbf{k} -algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with 1 generated by $1_{\mathbf{i}}$, \varkappa_l , σ_k , with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 0, 1, \dots, m-1$ modulo the following defining relations

$$\begin{aligned}
(a) \quad & 1_{\mathbf{i}} 1_{\mathbf{i}'} = \delta_{\mathbf{i}, \mathbf{i}'} 1_{\mathbf{i}}, \quad \sigma_k 1_{\mathbf{i}} = 1_{s_k \mathbf{i}} \sigma_k, \quad \varkappa_l 1_{\mathbf{i}} = 1_{\mathbf{i}} \varkappa_l, \\
(b) \quad & \varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l, \\
(c) \quad & \sigma_k^2 1_{\mathbf{i}} = Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) 1_{\mathbf{i}}, \\
(d) \quad & \sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k \text{ if } 1 \leq k < k' - 1 < m - 1 \text{ or } 0 = k < k' \neq 2, \\
(e) \quad & (\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_{\mathbf{i}} = \\
& = \begin{cases} \frac{Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) - Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_{s_k(k)+1})}{\varkappa_k - \varkappa_{s_k(k)+1}} 1_{\mathbf{i}} & \text{if } i_k = i_{s_k(k)+1}, \\ 0 & \text{else,} \end{cases} \\
(f) \quad & (\sigma_k \varkappa_l - \varkappa_{s_k(l)} \sigma_k) 1_{\mathbf{i}} = \begin{cases} -1_{\mathbf{i}} & \text{if } l = k, i_k = i_{s_k(k)}, \\ 1_{\mathbf{i}} & \text{if } l = s_k(k), i_k = i_{s_k(k)}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Here we have set $\varkappa_{1-l} = -\varkappa_l$ and

$$(1.1) \quad Q_{i,j}(u, v) = \begin{cases} (-1)^{h_{i,j}} (u - v)^{-i \cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases}$$

If $m = 0$ we set ${}^\circ\mathbf{R}(\Gamma)_0 = \mathbf{k} \oplus \mathbf{k}$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $\mathbf{i} = i\theta(i)$, and

$${}^\circ\mathbf{R}(\Gamma)_\nu = \mathbf{k}[\varkappa_1] 1_{\mathbf{i}} \oplus \mathbf{k}[\varkappa_1] 1_{\theta(\mathbf{i})}.$$

We'll abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_{\mathbf{i}}$ and $\varkappa_{\mathbf{i},l} = \varkappa_l 1_{\mathbf{i}}$. The grading on ${}^\circ\mathbf{R}(\Gamma)_0$ is the trivial one. For $m \geq 1$ the grading on ${}^\circ\mathbf{R}(\Gamma)_\nu$ is given by the following rules :

$$\begin{aligned}
\deg(1_{\mathbf{i}}) &= 0, \\
\deg(\varkappa_{\mathbf{i},l}) &= 2, \\
\deg(\sigma_{\mathbf{i},k}) &= -i_k \cdot i_{s_k(k)}.
\end{aligned}$$

We define ω to be the unique involution of the graded \mathbf{k} -algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ which fixes $1_{\mathbf{i}}$, \varkappa_l , σ_k . We set ω to be identity on ${}^\circ\mathbf{R}(\Gamma)_0$.

1.2. Relation with the graded \mathbf{k} -algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$. A family of graded \mathbf{k} -algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ was introduced in [VV, sec. 5.1], for λ an arbitrary dimension vector in \mathbf{NI} . Here we'll only consider the special case $\lambda = 0$, and we abbreviate ${}^\theta\mathbf{R}(\Gamma)_\nu = {}^\theta\mathbf{R}(\Gamma)_{0,\nu}$. Recall that if $\nu \neq 0$ then ${}^\theta\mathbf{R}(\Gamma)_\nu$ is the graded \mathbf{k} -algebra with 1 generated

by $1_{\mathbf{i}}, \varkappa_l, \sigma_k, \pi_1$, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 1, \dots, m-1$ such that $1_{\mathbf{i}}, \varkappa_l$ and σ_k satisfy the same relations as before and

$$\begin{aligned} \pi_1^2 &= 1, \quad \pi_1 1_{\mathbf{i}} \pi_1 = 1_{\varepsilon_1 \mathbf{i}}, \quad \pi_1 \varkappa_l \pi_1 = \varkappa_{\varepsilon_1(l)}, \\ (\pi_1 \sigma_1)^2 &= (\sigma_1 \pi_1)^2, \quad \pi_1 \sigma_k \pi_1 = \sigma_k \text{ if } k \neq 1. \end{aligned}$$

If $\nu = 0$ then ${}^\theta \mathbf{R}(\Gamma)_0 = \mathbf{k}$. The grading is given by setting $\deg(1_{\mathbf{i}})$, $\deg(\varkappa_{\mathbf{i},l})$, $\deg(\sigma_{\mathbf{i},k})$ to be as before and $\deg(\pi_1 1_{\mathbf{i}}) = 0$. In the rest of Section 1 we'll assume $m > 0$. Then there is a canonical inclusion of graded \mathbf{k} -algebras

$$(1.2) \quad {}^\circ \mathbf{R}(\Gamma)_\nu \subset {}^\theta \mathbf{R}(\Gamma)_\nu$$

such that $1_{\mathbf{i}}, \varkappa_l, \sigma_k \mapsto 1_{\mathbf{i}}, \varkappa_l, \sigma_k$ for $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, \dots, m$, $k = 1, \dots, m-1$ and such that $\sigma_0 \mapsto \pi_1 \sigma_1 \pi_1$. From now on we'll write $\sigma_0 = \pi_1 \sigma_1 \pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded \mathbf{k} -algebra ${}^\theta \mathbf{R}(\Gamma)_\nu$ which normalizes ${}^\circ \mathbf{R}(\Gamma)_\nu$. Thus it yields an involution

$$\gamma : {}^\circ \mathbf{R}(\Gamma)_\nu \rightarrow {}^\circ \mathbf{R}(\Gamma)_\nu.$$

Let $\langle \gamma \rangle$ be the group of two elements generated by γ . The smash product ${}^\circ \mathbf{R}(\Gamma)_\nu \rtimes \langle \gamma \rangle$ is a graded \mathbf{k} -algebra such that $\deg(\gamma) = 0$. There is an unique isomorphism of graded \mathbf{k} -algebras

$$(1.3) \quad {}^\circ \mathbf{R}(\Gamma)_\nu \rtimes \langle \gamma \rangle \rightarrow {}^\theta \mathbf{R}(\Gamma)_\nu$$

which is identity on ${}^\circ \mathbf{R}(\Gamma)_\nu$ and which takes γ to π_1 .

1.3. The polynomial representation and the PBW theorem. For any \mathbf{i} in ${}^\theta I^\nu$ let ${}^\theta \mathbf{F}_{\mathbf{i}}$ be the subalgebra of ${}^\circ \mathbf{R}(\Gamma)_\nu$ generated by $1_{\mathbf{i}}$ and $\varkappa_{\mathbf{i},l}$ with $l = 1, 2, \dots, m$. It is a polynomial algebra. Let

$${}^\theta \mathbf{F}_\nu = \bigoplus_{\mathbf{i} \in {}^\theta I^\nu} {}^\theta \mathbf{F}_{\mathbf{i}}.$$

The group W_m acts on ${}^\theta \mathbf{F}_\nu$ via $w(\varkappa_{\mathbf{i},l}) = \varkappa_{w(\mathbf{i}),w(l)}$ for any $w \in W_m$. Consider the fixed points set

$${}^\circ \mathbf{S}_\nu = ({}^\theta \mathbf{F}_\nu)^{{}^\circ W_m}.$$

Regard ${}^\theta \mathbf{R}(\Gamma)_\nu$ and $\text{End}({}^\theta \mathbf{F}_\nu)$ as ${}^\theta \mathbf{F}_\nu$ -algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded ${}^\theta \mathbf{F}_\nu$ -algebra morphism ${}^\theta \mathbf{R}(\Gamma)_\nu \rightarrow \text{End}({}^\theta \mathbf{F}_\nu)$. It restricts via (1.2) to an injective graded ${}^\theta \mathbf{F}_\nu$ -algebra morphism

$${}^\circ \mathbf{R}(\Gamma)_\nu \rightarrow \text{End}({}^\theta \mathbf{F}_\nu).$$

Next, recall that ${}^\circ W_m$ is the Weyl group of type D_m with simple reflections s_0, \dots, s_{m-1} . For each w in ${}^\circ W_m$ we choose a reduced decomposition \dot{w} of w . It has the following form

$$w = s_{k_1} s_{k_2} \cdots s_{k_r}, \quad 0 \leq k_1, k_2, \dots, k_r \leq m-1.$$

We define an element $\sigma_{\dot{w}}$ in ${}^\circ \mathbf{R}(\Gamma)_\nu$ by

$$(1.4) \quad \sigma_{\dot{w}} = \sum_{\mathbf{i}} 1_{\mathbf{i}} \sigma_{\dot{w}}, \quad 1_{\mathbf{i}} \sigma_{\dot{w}} = \begin{cases} 1_{\mathbf{i}} & \text{if } r = 0 \\ 1_{\mathbf{i}} \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_r} & \text{else,} \end{cases}$$

Observe that the element $\sigma_{\dot{w}}$ may depend on the choice of the reduced decomposition \dot{w} .

1.4. Proposition. *The \mathbf{k} -algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ is a free (left or right) ${}^\theta\mathbf{F}_\nu$ -module with basis $\{\sigma_{\dot{w}}; w \in {}^\circ W_m\}$. Its rank is $2^{m-1}m!$. The operator $1_{\mathbf{i}}\sigma_{\dot{w}}$ is homogeneous and its degree is independent of the choice of the reduced decomposition \dot{w} .*

Proof : The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with $1_{\mathbf{i}}, \varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$ in degree 1. The *Nil Hecke algebra* of type D_m is the \mathbf{k} -algebra ${}^\circ\mathbf{NH}_m$ generated by $\bar{\sigma}_0, \bar{\sigma}_1, \dots, \bar{\sigma}_{m-1}$ with relations

$$\bar{\sigma}_k \bar{\sigma}_{k'} = \bar{\sigma}_{k'} \bar{\sigma}_k \text{ if } 1 \leq k < k' - 1 < m - 1 \text{ or } 0 = k < k' \neq 2,$$

$$\bar{\sigma}_{s_k(k)} \bar{\sigma}_k \bar{\sigma}_{s_k(k)} = \bar{\sigma}_k \bar{\sigma}_{s_k(k)} \bar{\sigma}_k, \quad \bar{\sigma}_k^2 = 0.$$

We can form the semidirect product ${}^\theta\mathbf{F}_\nu \rtimes {}^\circ\mathbf{NH}_m$, which is generated by $1_{\mathbf{i}}, \bar{\varkappa}_l, \bar{\sigma}_k$ with the relations above and

$$\bar{\sigma}_k \bar{\varkappa}_l = \bar{\varkappa}_{s_k(l)} \bar{\sigma}_k, \quad \bar{\varkappa}_l \bar{\varkappa}_{l'} = \bar{\varkappa}_{l'} \bar{\varkappa}_l.$$

One proves as in [VV, prop. 5.5] that the map

$${}^\theta\mathbf{F}_\nu \rtimes {}^\circ\mathbf{NH}_m \rightarrow \text{gr}({}^\circ\mathbf{R}(\Gamma)_\nu), \quad 1_{\mathbf{i}} \mapsto 1_{\mathbf{i}}, \quad \bar{\varkappa}_l \mapsto \varkappa_l, \quad \bar{\sigma}_k \mapsto \sigma_k.$$

is an isomorphism of \mathbf{k} -algebras. □

Let ${}^\theta\mathbf{F}'_\nu = \bigoplus_{\mathbf{i}} {}^\theta\mathbf{F}'_{\mathbf{i}}$, where ${}^\theta\mathbf{F}'_{\mathbf{i}}$ is the localization of the ring ${}^\theta\mathbf{F}_{\mathbf{i}}$ with respect to the multiplicative system generated by

$$\{\varkappa_{\mathbf{i},l} \pm \varkappa_{\mathbf{i},l'}; 1 \leq l \neq l' \leq m\} \cup \{\varkappa_{\mathbf{i},l}; l = 1, 2, \dots, m\}.$$

1.5. Corollary. *The inclusion ${}^\circ\mathbf{R}(\Gamma)_\nu \subset \text{End}({}^\theta\mathbf{F}_\nu)$ yields an isomorphism of ${}^\theta\mathbf{F}'_\nu$ -algebras ${}^\theta\mathbf{F}'_\nu \otimes_{{}^\theta\mathbf{F}_\nu} {}^\circ\mathbf{R}(\Gamma)_\nu \rightarrow {}^\theta\mathbf{F}'_\nu \rtimes {}^\circ W_m$, such that for each \mathbf{i} and each $l = 1, 2, \dots, m, k = 0, 1, 2, \dots, m-1$ we have*

$$(1.5) \quad \begin{aligned} 1_{\mathbf{i}} &\mapsto 1_{\mathbf{i}}, \\ \varkappa_{\mathbf{i},l} &\mapsto \varkappa_l 1_{\mathbf{i}}, \\ \sigma_{\mathbf{i},k} &\mapsto \begin{cases} (\varkappa_k - \varkappa_{s_k(k)})^{-1} (s_k - 1) 1_{\mathbf{i}} & \text{if } i_k = i_{s_k(k)}, \\ (\varkappa_k - \varkappa_{s_k(k)})^{h_{i_{s_k(k)}, i_k}} s_k 1_{\mathbf{i}} & \text{if } i_k \neq i_{s_k(k)}. \end{cases} \end{aligned}$$

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4. □

Restricting the ${}^\theta\mathbf{F}_\nu$ -action on ${}^\circ\mathbf{R}(\Gamma)_\nu$ to the \mathbf{k} -subalgebra ${}^\circ\mathbf{S}_\nu$ we get a structure of graded ${}^\circ\mathbf{S}_\nu$ -algebra on ${}^\circ\mathbf{R}(\Gamma)_\nu$.

1.6. Proposition. (a) ${}^\circ\mathbf{S}_\nu$ is isomorphic to the center of ${}^\circ\mathbf{R}(\Gamma)_\nu$.

(b) ${}^\circ\mathbf{R}(\Gamma)_\nu$ is a free graded module over ${}^\circ\mathbf{S}_\nu$ of rank $(2^{m-1}m!)^2$.

Proof : Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4. □

2. AFFINE HECKE ALGEBRAS OF TYPE D

2.1. Affine Hecke algebras of type D. Fix p in \mathbf{k}^\times . For any integer $m \geq 0$ we define the extended affine Hecke algebra \mathbf{H}_m of type D_m as follows. If $m > 1$ then \mathbf{H}_m is the \mathbf{k} -algebra with 1 generated by

$$T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m$$

satisfying the following defining relations :

- (a) $X_l X_{l'} = X_{l'} X_l$,
- (b) $T_k T_{s_k(k)} T_k = T_{s_k(k)} T_k T_{s_k(k)}$, $T_k T_{k'} = T_{k'} T_k$ if $1 \leq k < k' - 1$ or $k = 0, k' \neq 2$,
- (c) $(T_k - p)(T_k + p^{-1}) = 0$,
- (d) $T_0 X_1^{-1} T_0 = X_2$, $T_k X_k T_k = X_{s_k(k)}$ if $k \neq 0$, $T_k X_l = X_l T_k$ if $k \neq 0, l, l-1$ or $k = 0, l \neq 1, 2$.

Finally, we set $\mathbf{H}_0 = \mathbf{k} \oplus \mathbf{k}$ and $\mathbf{H}_1 = \mathbf{k}[X_1^{\pm 1}]$.

2.2. Remarks. (a) The extended affine Hecke algebra \mathbf{H}_m^B of type B_m with parameters $p, q \in \mathbf{k}^\times$ such that $q = 1$ is generated by $P, T_k, X_l^{\pm 1}, k = 1, \dots, m-1, l = 1, \dots, m$ such that $T_k, X_l^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \quad (PT_1)^2 = (T_1 P)^2, \quad PT_k = T_k P \text{ if } k \neq 1,$$

$$PX_1^{-1}P = X_1, \quad PX_l = X_l P \text{ if } l \neq 1.$$

See e.g., [VV, sec. 6.1]. There is an obvious \mathbf{k} -algebra embedding $\mathbf{H}_m \subset \mathbf{H}_m^B$. Let γ denote also the involution $\mathbf{H}_m \rightarrow \mathbf{H}_m, a \mapsto PaP$. We have a canonical isomorphism of \mathbf{k} -algebras

$$\mathbf{H}_m \rtimes \langle \gamma \rangle \simeq \mathbf{H}_m^B.$$

Compare Section 1.2.

(b) Given a connected reductive group G we call *affine Hecke algebra of G* the Hecke algebra of the extended affine Weyl group $W \ltimes P$, where W is the Weyl group of (G, T) , P is the group of characters of T , and T is a maximal torus of G . Then \mathbf{H}_m is the affine Hecke algebra of the group $SO(2m)$. Let \mathbf{H}_m^e be the affine Hecke algebra of the group $Spin(2m)$. It is generated by \mathbf{H}_m and an element X_0 such that

$$X_0^2 = X_1 X_2 \dots X_m, \quad T_k X_0 = X_0 T_k \text{ if } k \neq 0, \quad T_0 X_0 X_1^{-1} X_2^{-1} T_0 = X_0.$$

Thus \mathbf{H}_m is the fixed point subset of the \mathbf{k} -algebra automorphism of \mathbf{H}_m^e taking T_k, X_l to $T_k, (-1)^{\delta_{l,0}} X_l$ for all k, l . Therefore, if p is not a root of 1 the simple \mathbf{H}_m -modules can be recovered from the Kazhdan-Lusztig classification of the simple \mathbf{H}_m^e -modules via Clifford theory, see e.g., [Re].

2.3. Intertwiners and blocks of \mathbf{H}_m . We define

$$\mathbf{A} = \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}], \quad \mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_l X_{l'}^{\pm 1}, \quad 1 - p^2 X_l^{\pm 1} X_{l'}^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k is the element of \mathbf{H}'_m given by the following formulas

$$(2.1) \quad \varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}} (T_k - p).$$

The group ${}^\circ W_m$ acts on \mathbf{A}' as follows

$$(s_k a)(X_1, \dots, X_m) = a(X_1, \dots, X_{k+1}, X_k, \dots, X_m) \text{ if } k \neq 0,$$

$$(s_0 a)(X_1, \dots, X_m) = a(X_2^{-1}, X_1^{-1}, \dots, X_m).$$

There is an isomorphism of \mathbf{A}' -algebras

$$\mathbf{A}' \rtimes {}^\circ W_m \rightarrow \mathbf{H}'_m, \quad s_k \mapsto \varphi_k.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z} \rtimes \{-1, 1\}$ acts on \mathbf{k}^\times by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$ -invariant subset I of \mathbf{k}^\times we denote by $\mathbf{H}_m\text{-Mod}_I$ the category of all \mathbf{H}_m -modules such that the action of X_1, X_2, \dots, X_m is locally finite with eigenvalues in I . We associate to the set I and to the element $p \in \mathbf{k}^\times$ a quiver Γ as follows. The set of vertices is I , and there is one arrow $p^2 i \rightarrow i$ whenever i lies in I . We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $p^2 i \rightarrow i$ to the arrow $i^{-1} \rightarrow p^{-2} i^{-1}$. We'll assume that the set I does not contain 1 nor -1 and that $p \neq 1, -1$. Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

2.4. Remark. We may assume that $I = \pm\{p^n; n \in \mathbb{Z}_{\text{odd}}\}$. See the discussion in [KM]. Then Γ is of type A_∞ if p has infinite order and Γ is of type $A_r^{(1)}$ if p^2 is a primitive r -th root of unity.

2.5. \mathbf{H}_m -modules versus ${}^\circ \mathbf{R}_m$ -modules. Assume that $m \geq 1$. We define the graded \mathbf{k} -algebra

$$\begin{aligned} {}^\theta \mathbf{R}_{I,m} &= \bigoplus_{\nu} {}^\theta \mathbf{R}_{I,\nu}, \quad {}^\theta \mathbf{R}_{I,\nu} = {}^\theta \mathbf{R}(\Gamma)_\nu, \quad {}^\circ \mathbf{R}_{I,m} = \bigoplus_{\nu} {}^\circ \mathbf{R}_{I,\nu}, \quad {}^\circ \mathbf{R}_{I,\nu} = {}^\circ \mathbf{R}(\Gamma)_\nu, \\ \theta I^m &= \bigsqcup_{\nu} \theta I^\nu, \end{aligned}$$

where ν runs over the set of all dimension vectors in ${}^\theta \mathbb{N}I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate

$${}^\theta \mathbf{R}_\nu = {}^\theta \mathbf{R}_{I,\nu}, \quad {}^\theta \mathbf{R}_m = {}^\theta \mathbf{R}_{I,m}, \quad {}^\circ \mathbf{R}_\nu = {}^\circ \mathbf{R}_{I,\nu}, \quad {}^\circ \mathbf{R}_m = {}^\circ \mathbf{R}_{I,m}.$$

Note that ${}^\theta \mathbf{R}_\nu$ and ${}^\theta \mathbf{R}_m$ are the same as in [VV, sec. 6.4], with $\lambda = 0$. Note also that the \mathbf{k} -algebra ${}^\circ \mathbf{R}_m$ may not have 1, because the set I may be infinite. We define ${}^\circ \mathbf{R}_m\text{-Mod}_0$ as the category of all (non-graded) ${}^\circ \mathbf{R}_m$ -modules such that the elements $\varkappa_1, \varkappa_2, \dots, \varkappa_m$ act locally nilpotently. Let ${}^\circ \mathbf{R}_m\text{-fMod}_0$ and $\mathbf{H}_m\text{-fMod}_I$ be the full subcategories of finite dimensional modules in ${}^\circ \mathbf{R}_m\text{-Mod}_0$ and $\mathbf{H}_m\text{-Mod}_I$ respectively. Fix a formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo (\varkappa^2) .

2.6. Theorem. *We have an equivalence of categories*

$${}^{\circ}\mathbf{R}_m\text{-}\mathbf{Mod}_0 \rightarrow \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

- (a) X_l acts on $1_{\mathbf{i}}M$ by $i_l^{-1}f(\varkappa_l)$ for each $l = 1, 2, \dots, m$,
- (b) if $m > 1$ then T_k acts on $1_{\mathbf{i}}M$ as follows for each $k = 0, 1, \dots, m-1$,

$$\begin{aligned} & \frac{(pf(\varkappa_k) - p^{-1}f(\varkappa_{s_k(k)}))(\varkappa_k - \varkappa_{s_k(k)})}{f(\varkappa_k) - f(\varkappa_{s_k(k)})} \sigma_k + p & \text{if } i_{s_k(k)} = i_k, \\ & \frac{f(\varkappa_k) - f(\varkappa_{s_k(k)})}{(p^{-1}f(\varkappa_k) - pf(\varkappa_{s_k(k)}))(\varkappa_k - \varkappa_{s_k(k)})} \sigma_k + \frac{(p^{-2} - 1)f(\varkappa_{s_k(k)})}{pf(\varkappa_k) - p^{-1}f(\varkappa_{s_k(k)})} & \text{if } i_{s_k(k)} = p^2 i_k, \\ & \frac{pi_k f(\varkappa_k) - p^{-1}i_{s_k(k)} f(\varkappa_{s_k(k)})}{i_k f(\varkappa_k) - i_{s_k(k)} f(\varkappa_{s_k(k)})} \sigma_k + \frac{(p^{-1} - p)i_k f(\varkappa_{s_k(k)})}{i_{s_k(k)} f(\varkappa_k) - i_k f(\varkappa_{s_k(k)})} & \text{if } i_{s_k(k)} \neq i_k, p^2 i_k. \end{aligned}$$

Proof : This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1). \square

2.7. Corollary. *There is an equivalence of categories*

$$\Psi : {}^{\circ}\mathbf{R}_m\text{-}\mathbf{fMod}_0 \rightarrow \mathbf{H}_m\text{-}\mathbf{fMod}_I, \quad M \mapsto M.$$

2.8. Remarks. (a) Let \mathfrak{g} be the Lie algebra of $G = SO(2m)$. Fix a maximal torus $T \subset G$. The group of characters of T is the lattice $\bigoplus_{l=1}^m \mathbb{Z} \varepsilon_l$, with Bourbaki's notation. Fix a dimension vector $\nu \in {}^{\theta}\mathbb{N}I$. Recall the sequence $\mathbf{i}_e = (i_{1-m}, \dots, i_{m-1}, i_m)$ from Section 0.3. Let $g \in T$ be the element such that $\varepsilon_l(g) = i_l^{-1}$ for each $l = 1, 2, \dots, m$. Recall also the notation ${}^{\theta}\mathbf{V}_{\nu}$, \mathbf{V} , ${}^{\theta}E_{\mathbf{V}}$, and ${}^{\theta}G_{\mathbf{V}}$ from [VV]. Then \mathbf{V} is an object of ${}^{\theta}\mathbf{V}_{\nu}$, ${}^{\theta}G_{\mathbf{V}} = G_g$ is the centralizer of g in G , and

$${}^{\theta}E_{\mathbf{V}} = \mathfrak{g}_{g,p}, \quad \mathfrak{g}_{g,p} = \{x \in \mathfrak{g}; \text{ad}_g(x) = p^2 x\}.$$

Let F_g be the set of all Borel Lie subalgebras of \mathfrak{g} fixed by the adjoint action of g . It is a non connected manifold whose connected components are labelled by ${}^{\theta}I_{+}^{\nu}$. In Section 3.14 we'll introduce two central idempotents $1_{\nu,+}$, $1_{\nu,-}$ of ${}^{\circ}\mathbf{R}_{\nu}$. This yields a graded \mathbf{k} -algebra decomposition

$${}^{\circ}\mathbf{R}_{\nu} = {}^{\circ}\mathbf{R}_{\nu}1_{\nu,+} \oplus {}^{\circ}\mathbf{R}_{\nu}1_{\nu,-}.$$

By [VV, thm. 5.8] the graded \mathbf{k} -algebra ${}^{\circ}\mathbf{R}_{\nu}1_{\nu,+}$ is isomorphic to

$$\text{Ext}_{G_g}^*(\mathcal{L}_{g,p}, \mathcal{L}_{g,p}),$$

where $\mathcal{L}_{g,p}$ is the direct image of the constant perverse sheaf by the projection

$$\{(\mathfrak{b}, x) \in F_g \times \mathfrak{g}_{g,p}; x \in \mathfrak{b}\} \rightarrow \mathfrak{g}_{g,p}, \quad (\mathfrak{b}, x) \mapsto x.$$

The complex $\mathcal{L}_{g,p}$ has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if \mathbf{k} is any field. Set $f(\varkappa) = 1 + \varkappa$ for instance.

2.9. Induction and restriction of \mathbf{H}_m -modules. For $i \in I$ we define functors

$$(2.2) \quad \begin{aligned} E_i &: \mathbf{H}_{m+1}\text{-fMod}_I \rightarrow \mathbf{H}_m\text{-fMod}_I, \\ F_i &: \mathbf{H}_m\text{-fMod}_I \rightarrow \mathbf{H}_{m+1}\text{-fMod}_I, \end{aligned}$$

where $E_i M \subset M$ is the generalized i^{-1} -eigenspace of the X_{m+1} -action, and where

$$F_i M = \text{Ind}_{\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]}^{\mathbf{H}_{m+1}}(M \otimes \mathbf{k}_i).$$

Here \mathbf{k}_i is the 1-dimensional representation of $\mathbf{k}[X_{m+1}^{\pm 1}]$ defined by $X_{m+1} \mapsto i^{-1}$.

3. GLOBAL BASES OF ${}^\circ \mathbf{V}$ AND PROJECTIVE GRADED ${}^\circ \mathbf{R}$ -MODULES

3.1. The Grothendieck groups of ${}^\circ \mathbf{R}_m$. The graded \mathbf{k} -algebra ${}^\circ \mathbf{R}_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded \mathbf{k} -subalgebra. Therefore any simple object of ${}^\circ \mathbf{R}_m\text{-mod}$ is finite-dimensional and there is a finite number of isomorphism classes of simple modules in ${}^\circ \mathbf{R}_m\text{-mod}$. The Abelian group $G({}^\circ \mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of ${}^\circ \mathbf{R}_m\text{-mod}$. The Abelian group $K({}^\circ \mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G({}^\circ \mathbf{R}_m)$ and $K({}^\circ \mathbf{R}_m)$ are free \mathcal{A} -modules, where v shifts the grading by 1. We consider the following \mathcal{A} -modules

$$\begin{aligned} {}^\circ \mathbf{K}_I &= \bigoplus_{m \geq 0} {}^\circ \mathbf{K}_{I,m}, & {}^\circ \mathbf{K}_{I,m} &= K({}^\circ \mathbf{R}_m), \\ {}^\circ \mathbf{G}_I &= \bigoplus_{m \geq 0} {}^\circ \mathbf{G}_{I,m}, & {}^\circ \mathbf{G}_{I,m} &= G({}^\circ \mathbf{R}_m). \end{aligned}$$

We'll also abbreviate

$${}^\circ \mathbf{K}_{I,*} = \bigoplus_{m > 0} {}^\circ \mathbf{K}_{I,m}, \quad {}^\circ \mathbf{G}_{I,*} = \bigoplus_{m > 0} {}^\circ \mathbf{G}_{I,m}.$$

From now on, to unburden the notation we may abbreviate ${}^\circ \mathbf{R} = {}^\circ \mathbf{R}_m$, hoping it will not create any confusion. For any M, N in ${}^\circ \mathbf{R}\text{-mod}$ we set

$$(M : N) = \text{gdim}(M^\omega \otimes_{{}^\circ \mathbf{R}} N), \quad \langle M : N \rangle = \text{gdim} \text{hom}_{{}^\circ \mathbf{R}}(M, N),$$

where ω is the involution defined in Section 1.1. The Cartan pairing is the perfect \mathcal{A} -bilinear form

$${}^\circ \mathbf{K}_I \times {}^\circ \mathbf{G}_I \rightarrow \mathcal{A}, \quad (P, M) \mapsto \langle P : M \rangle.$$

First, we concentrate on the \mathcal{A} -module ${}^\circ \mathbf{G}_I$. Consider the duality

$${}^\circ \mathbf{R}\text{-fmod} \rightarrow {}^\circ \mathbf{R}\text{-fmod}, \quad M \mapsto M^b = \text{hom}(M, \mathbf{k}),$$

with the action and the grading given by

$$(xf)(m) = f(\omega(x)m), \quad (M^b)_d = \text{Hom}(M_{-d}, \mathbf{k}).$$

This duality functor yields an \mathcal{A} -antilinear map

$${}^\circ \mathbf{G}_I \rightarrow {}^\circ \mathbf{G}_I, \quad M \mapsto M^b.$$

Let ${}^\circ B$ denote the set of isomorphism classes of simple objects of ${}^\circ \mathbf{R}\text{-fMod}_0$. We can now define the upper global basis of ${}^\circ \mathbf{G}_I$ as follows. The proof is given in Section 3.21.

3.2. Proposition/Definition. *For each b in ${}^\circ B$ there is a unique selfdual irreducible graded ${}^\circ \mathbf{R}$ -module ${}^\circ G^{\text{up}}(b)$ which is isomorphic to b as a (non graded) ${}^\circ \mathbf{R}$ -module. We set ${}^\circ G^{\text{up}}(0) = 0$ and ${}^\circ \mathbf{G}^{\text{up}} = \{{}^\circ G^{\text{up}}(b); b \in {}^\circ B\}$. Hence ${}^\circ \mathbf{G}^{\text{up}}$ is a \mathcal{A} -basis of ${}^\circ \mathbf{G}_I$.*

Now, we concentrate on the \mathcal{A} -module ${}^\circ \mathbf{K}_I$. We equip ${}^\circ \mathbf{K}_I$ with the symmetric \mathcal{A} -bilinear form

$$(3.1) \quad {}^\circ \mathbf{K}_I \times {}^\circ \mathbf{K}_I \rightarrow \mathcal{A}, \quad (M, N) \mapsto (M : N).$$

Consider the duality

$${}^\circ \mathbf{R}\text{-proj} \rightarrow {}^\circ \mathbf{R}\text{-proj}, \quad P \mapsto P^\sharp = \text{hom}_{{}^\circ \mathbf{R}}(P, {}^\circ \mathbf{R}),$$

with the action and the grading given by

$$(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \text{Hom}_{{}^\circ \mathbf{R}}(P[-d], {}^\circ \mathbf{R}).$$

This duality functor yields an \mathcal{A} -antilinear map

$${}^\circ \mathbf{K}_I \rightarrow {}^\circ \mathbf{K}_I, \quad P \mapsto P^\sharp.$$

Set $\mathcal{K} = \mathbb{Q}(v)$. Let $\mathcal{K} \rightarrow \mathcal{K}$, $f \mapsto \bar{f}$ be the unique involution such that $\bar{v} = v^{-1}$.

3.3. Definition. *For each b in ${}^\circ B$ let ${}^\circ G^{\text{low}}(b)$ be the unique indecomposable graded module in ${}^\circ \mathbf{R}\text{-proj}$ whose top is isomorphic to ${}^\circ G^{\text{up}}(b)$. We set ${}^\circ G^{\text{low}}(0) = 0$ and ${}^\circ \mathbf{G}^{\text{low}} = \{{}^\circ G^{\text{low}}(b); b \in {}^\circ B\}$, a \mathcal{A} -basis of ${}^\circ \mathbf{K}_I$.*

3.4. Proposition. (a) *We have $\langle {}^\circ G^{\text{low}}(b) : {}^\circ G^{\text{up}}(b') \rangle = \delta_{b,b'}$ for each b, b' in ${}^\circ B$.*

(b) *We have $\langle P^\sharp : M \rangle = \overline{\langle P : M^\flat \rangle}$ for each P, M .*

(c) *We have ${}^\circ G^{\text{low}}(b)^\sharp = {}^\circ G^{\text{low}}(b)$ for each b in ${}^\circ B$.*

The proof is the same as in [VV, prop. 8.4].

3.5. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Consider the graded ${}^\circ \mathbf{R}_\nu$ -modules

$${}^\circ \mathbf{R}_\mathbf{i} = {}^\circ \mathbf{R}1_\mathbf{i} = 1_\mathbf{i} {}^\circ \mathbf{R}, \quad {}^\circ \mathbf{L}_\mathbf{i} = \text{top}({}^\circ \mathbf{R}_\mathbf{i}).$$

The global bases are given by

$${}^\circ \mathbf{G}_\nu^{\text{low}} = \{{}^\circ \mathbf{R}_\mathbf{i}, {}^\circ \mathbf{R}_{\theta(\mathbf{i})}\}, \quad {}^\circ \mathbf{G}_\nu^{\text{up}} = \{{}^\circ \mathbf{L}_\mathbf{i}, {}^\circ \mathbf{L}_{\theta(\mathbf{i})}\}.$$

For $m = 0$ we have ${}^\circ \mathbf{R}_0 = \mathbf{k} \oplus \mathbf{k}$. Set $\phi_+ = \mathbf{k} \oplus 0$ and $\phi_- = 0 \oplus \mathbf{k}$. We have

$${}^\circ \mathbf{G}_0^{\text{low}} = {}^\circ \mathbf{G}_0^{\text{up}} = \{\phi_+, \phi_-\}.$$

3.6. Definition of the operators e_i, f_i, e'_i, f'_i . In this section we'll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in ${}^\circ W_{m+1}$ of the cosets in ${}^\circ W_m \backslash {}^\circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$

For each element w of $D_{m,1;m,1}$ we set

$$W(w) = {}^\circ W_m \cap w({}^\circ W_m)w^{-1}.$$

Let \mathbf{R}_1 be the \mathbf{k} -algebra generated by elements $1_i, \varkappa_i, i \in I$, satisfying the defining relations $1_i 1_{i'} = \delta_{i,i'} 1_i$ and $\varkappa_i = 1_i \varkappa_i 1_i$. We equip \mathbf{R}_1 with the grading such that $\deg(1_i) = 0$ and $\deg(\varkappa_i) = 2$. Let

$$\mathbf{R}_i = 1_i \mathbf{R}_1 = \mathbf{R}_1 1_i, \quad \mathbf{L}_i = \text{top}(\mathbf{R}_i) = \mathbf{R}_i / (\varkappa_i).$$

Then \mathbf{R}_i is a graded projective \mathbf{R}_1 -module and \mathbf{L}_i is simple. We abbreviate

$${}^\theta \mathbf{R}_{m,1} = {}^\theta \mathbf{R}_m \otimes \mathbf{R}_1, \quad {}^\circ \mathbf{R}_{m,1} = {}^\circ \mathbf{R}_m \otimes \mathbf{R}_1.$$

There is an unique inclusion of graded \mathbf{k} -algebras

$$(3.2) \quad \begin{aligned} {}^\theta \mathbf{R}_{m,1} &\rightarrow {}^\theta \mathbf{R}_{m+1}, \\ 1_{\mathbf{i}} \otimes 1_i &\mapsto 1_{\mathbf{i}'}, \\ 1_{\mathbf{i}} \otimes \varkappa_{i,l} &\mapsto \varkappa_{\mathbf{i}',m+l}, \\ \varkappa_{\mathbf{i},l} \otimes 1_i &\mapsto \varkappa_{\mathbf{i}',l}, \\ \pi_{\mathbf{i},1} \otimes 1_i &\mapsto \pi_{\mathbf{i}',1}, \\ \sigma_{\mathbf{i},k} \otimes 1_i &\mapsto \sigma_{\mathbf{i}',k}, \end{aligned}$$

where, given $\mathbf{i} \in {}^\theta I^m$ and $i \in I$, we have set $\mathbf{i}' = \theta(i)\mathbf{i}$, a sequence in ${}^\theta I^{m+1}$. This inclusion restricts to an inclusion ${}^\circ \mathbf{R}_{m,1} \subset {}^\circ \mathbf{R}_{m+1}$.

3.7. Lemma. *The graded ${}^\circ \mathbf{R}_{m,1}$ -module ${}^\circ \mathbf{R}_{m+1}$ is free of rank $2(m+1)$.*

Proof: For each w in $D_{m,1}$ we have the element $\sigma_{\dot{w}}$ in ${}^\circ \mathbf{R}_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

$${}^\circ \mathbf{R}_{m+1} = \bigoplus_{w \in D_{m,1}} {}^\circ \mathbf{R}_{m,1} \sigma_{\dot{w}}.$$

□

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$\psi^* : {}^\circ \mathbf{R}_{m+1}\text{-mod} \rightarrow {}^\circ \mathbf{R}_m\text{-mod} \times \mathbf{R}_1\text{-mod}$$

is the restriction and $\psi_!$, ψ_* are given by

$$\begin{aligned} \psi_! : & \begin{cases} {}^\circ\mathbf{R}_m\text{-mod} \times \mathbf{R}_1\text{-mod} \rightarrow {}^\circ\mathbf{R}_{m+1}\text{-mod}, \\ (M, M') \mapsto {}^\circ\mathbf{R}_{m+1} \otimes_{{}^\circ\mathbf{R}_{m,1}} (M \otimes M'), \end{cases} \\ \psi_* : & \begin{cases} {}^\circ\mathbf{R}_m\text{-mod} \times \mathbf{R}_1\text{-mod} \rightarrow {}^\circ\mathbf{R}_{m+1}\text{-mod}, \\ (M, M') \mapsto \text{hom}_{{}^\circ\mathbf{R}_{m,1}}({}^\circ\mathbf{R}_{m+1}, M \otimes M'). \end{cases} \end{aligned}$$

First, note that the functors $\psi_!$, ψ^* , ψ_* commute with the shift of the grading. Next, the functor ψ^* is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded ${}^\circ\mathbf{R}_{m,1}$ -module ${}^\circ\mathbf{R}_{m+1}$ is free of finite rank. Thus $\psi_!$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded ${}^\circ\mathbf{R}_{m,1}$ -module ${}^\circ\mathbf{R}_{m+1}$ is also free of finite rank. Thus the functor ψ_* is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further $\psi_!$ and ψ^* take projective graded modules to projective ones, because they are left adjoint to the exact functors ψ^* , ψ_* respectively. To summarize, the functors $\psi_!$, ψ^* , ψ_* are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\psi_!$, ψ^* take projective graded modules to projective ones.

For any graded ${}^\circ\mathbf{R}_m$ -module M we write

$$(3.3) \quad \begin{aligned} f_i(M) &= {}^\circ\mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \\ e_i(M) &= {}^\circ\mathbf{R}_{m-1} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} 1_{m-1,i} M. \end{aligned}$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$1_{m-1,i} M = \bigoplus_{\mathbf{i}} 1_{\theta(\mathbf{i})\mathbf{i}} M, \quad \mathbf{i} \in \theta I^{m-1}.$$

Note that $f_i(M)$ is a graded ${}^\circ\mathbf{R}_{m+1}$ -module, while $e_i(M)$ is a graded ${}^\circ\mathbf{R}_{m-1}$ -module. The tensor product in the definition of $e_i(M)$ is relative to the graded \mathbf{k} -algebra homomorphism

$${}^\circ\mathbf{R}_{m-1,1} \rightarrow {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_1 \rightarrow {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i \rightarrow {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{L}_i = {}^\circ\mathbf{R}_{m-1}.$$

In other words, let $e'_i(M)$ be the graded ${}^\circ\mathbf{R}_{m-1}$ -module obtained by taking the direct summand $1_{m-1,i} M$ and restricting it to ${}^\circ\mathbf{R}_{m-1}$. Observe that if M is finitely generated then $e'_i(M)$ may not lie in ${}^\circ\mathbf{R}_{m-1}\text{-mod}$. To remedy this, since $e'_i(M)$ affords a ${}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i$ -action we consider the graded ${}^\circ\mathbf{R}_{m-1}$ -module

$$e_i(M) = e'_i(M) / \varkappa_i e'_i(M).$$

3.8. Definition. *The functors e_i , f_i preserve the category ${}^\circ\mathbf{R}\text{-proj}$, yielding \mathcal{A} -linear operators on ${}^\circ\mathbf{K}_I$ which act on ${}^\circ\mathbf{K}_{I,*}$ by the formula (3.3) and satisfy also*

$$f_i(\phi_+) = {}^\circ\mathbf{R}_{\theta(i)i}, \quad f_i(\phi_-) = {}^\circ\mathbf{R}_{i\theta(i)}, \quad e_i(\mathbf{R}_{\theta(j)j}) = \delta_{i,j} \phi_+ + \delta_{i,\theta(j)} \phi_-.$$

Let e_i , f_i denote also the \mathcal{A} -linear operators on ${}^\circ\mathbf{G}_I$ which are the transpose of f_i , e_i with respect to the Cartan pairing.

Note that the symbols $e_i(M)$, $f_i(M)$ have a different meaning if M is viewed as an element of ${}^\circ\mathbf{K}_I$ or if M is viewed as an element of ${}^\circ\mathbf{G}_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].

3.9. Lemma. (a) The operators e_i, f_i on ${}^\circ\mathbf{G}_I$ are given by

$$e_i(M) = 1_{m-1,i}M \quad f_i(M) = \text{hom}_{{}^\circ\mathbf{R}_{m+1}}({}^\circ\mathbf{R}_{m+1}, M \otimes \mathbf{L}_i), \quad M \in {}^\circ\mathbf{R}_m\text{-fmod}.$$

(b) For each $M \in {}^\circ\mathbf{R}_m\text{-mod}$, $M' \in {}^\circ\mathbf{R}_{m+1}\text{-mod}$ we have

$$(e'_i(M') : M) = (M' : f_i(M)).$$

(c) We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\circ\mathbf{R}\text{-proj}$.

(d) We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\circ\mathbf{R}\text{-fmod}$.

3.10. Induction of \mathbf{H}_m -modules versus induction of ${}^\circ\mathbf{R}_m$ -modules. Recall the functors E_i, F_i on $\mathbf{H}\text{-fMod}_I$ defined in (2.2). We have also the functors

$$\mathbf{for} : {}^\circ\mathbf{R}_m\text{-fmod} \rightarrow {}^\circ\mathbf{R}_m\text{-fMod}_0, \quad \Psi : {}^\circ\mathbf{R}_m\text{-fMod}_0 \rightarrow \mathbf{H}_m\text{-fMod}_I,$$

where \mathbf{for} is the forgetting of the grading. Finally we define functors

$$(3.4) \quad \begin{aligned} E_i : {}^\circ\mathbf{R}_m\text{-fMod}_0 &\rightarrow {}^\circ\mathbf{R}_{m-1}\text{-fMod}_0, & E_i M &= 1_{m-1,i}M, \\ F_i : {}^\circ\mathbf{R}_m\text{-fMod}_0 &\rightarrow {}^\circ\mathbf{R}_{m+1}\text{-fMod}_0, & F_i M &= \psi_!(M, \mathbf{L}_i). \end{aligned}$$

3.11. Proposition. There are canonical isomorphisms of functors

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \mathbf{for} = \mathbf{for} \circ e_i, \quad F_i \circ \mathbf{for} = \mathbf{for} \circ f_{\theta(i)}.$$

Proof: The proof is the same as in [VV, prop. 8.17]. □

3.12. Proposition. (a) The functor Ψ yields an isomorphism of Abelian groups

$$\bigoplus_{m \geq 0} [{}^\circ\mathbf{R}_m\text{-fMod}_0] = \bigoplus_{m \geq 0} [\mathbf{H}_m\text{-fMod}_I].$$

The functors E_i, F_i yield endomorphisms of both sides which are intertwined by Ψ .

(b) The functor \mathbf{for} factors to a group isomorphism

$${}^\circ\mathbf{G}_I / (v - 1) = \bigoplus_{m \geq 0} [{}^\circ\mathbf{R}_m\text{-fMod}_0].$$

Proof: Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2. □

3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let ${}^\theta \mathbf{K}$, ${}^\theta B$, ${}^\theta G^{\text{low}}$, etc, denote the type B analogues of ${}^\circ \mathbf{K}$, ${}^\circ B$, ${}^\circ G^{\text{low}}$, etc. See [VV] for details. We'll use the same notation for the functors ψ^* , $\psi_!$, ψ_* , e_i , f_i , etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in {}^\theta \mathbf{NI}$ such that $|\nu| = 2m$. The inclusion of graded \mathbf{k} -algebras ${}^\circ \mathbf{R}_\nu \subset {}^\theta \mathbf{R}_\nu$ in (1.2) yields a restriction functor

$$\text{res} : {}^\theta \mathbf{R}_\nu\text{-mod} \rightarrow {}^\circ \mathbf{R}_\nu\text{-mod}$$

and an induction functor

$$\text{ind} : {}^\circ \mathbf{R}_\nu\text{-mod} \rightarrow {}^\theta \mathbf{R}_\nu\text{-mod}, \quad M \mapsto {}^\theta \mathbf{R}_\nu \otimes_{{}^\circ \mathbf{R}_\nu} M.$$

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of \mathcal{A} -modules

$$\text{res} : {}^\theta \mathbf{K}_{I,m} \rightarrow {}^\circ \mathbf{K}_{I,m}, \quad \text{res} : {}^\theta \mathbf{G}_{I,m} \rightarrow {}^\circ \mathbf{G}_{I,m},$$

$$\text{ind} : {}^\circ \mathbf{K}_{I,m} \rightarrow {}^\theta \mathbf{K}_{I,m}, \quad \text{ind} : {}^\circ \mathbf{G}_{I,m} \rightarrow {}^\theta \mathbf{G}_{I,m}.$$

Moreover, for any $P \in {}^\theta \mathbf{K}_{I,m}$ and any $L \in {}^\theta \mathbf{G}_{I,m}$ we have

$$(3.5) \quad \begin{aligned} \text{res}(P^\sharp) &= (\text{res}P)^\sharp, & \text{ind}(P^\sharp) &= (\text{ind}P)^\sharp \\ \text{res}(L^\flat) &= (\text{res}L)^\flat, & \text{ind}(L^\flat) &= (\text{ind}L)^\flat. \end{aligned}$$

Note also that ind and res are left and right adjoint functors, because

$${}^\theta \mathbf{R}_\nu \otimes_{{}^\circ \mathbf{R}_\nu} M = \text{hom}_{{}^\circ \mathbf{R}_\nu}({}^\theta \mathbf{R}_\nu, M)$$

as graded ${}^\theta \mathbf{R}_\nu$ -modules.

3.14. Definition. For any graded ${}^\circ \mathbf{R}_\nu$ -module M we define the graded ${}^\circ \mathbf{R}_\nu$ -module M^γ with the same underlying graded \mathbf{k} -vector space as M such that the action of ${}^\circ \mathbf{R}_\nu$ is twisted by γ , i.e., the graded \mathbf{k} -algebra ${}^\circ \mathbf{R}_\nu$ acts on M^γ by $a m = \gamma(a)m$ for $a \in {}^\circ \mathbf{R}_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that M^γ is simple (resp. projective, indecomposable) if M has the same property.

For any graded ${}^\circ \mathbf{R}_m$ -module M we have canonical isomorphisms of ${}^\circ \mathbf{R}$ -modules

$$(f_i(M))^\gamma = f_i(M^\gamma), \quad (e_i(M))^\gamma = e_i(M^\gamma).$$

The first isomorphism is given by

$${}^\circ \mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ \mathbf{R}_m} M \rightarrow {}^\circ \mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ \mathbf{R}_m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$

The second one is the identity map on the vector space $1_{m,i}M$.

Recall that ${}^\theta I^\nu$ is the disjoint union of ${}^\theta I_+^\nu$ and ${}^\theta I_-^\nu$. We set

$$1_{\nu,+} = \sum_{\mathbf{i} \in {}^\theta I_+^\nu} 1_{\mathbf{i}}, \quad 1_{\nu,-} = \sum_{\mathbf{i} \in {}^\theta I_-^\nu} 1_{\mathbf{i}}.$$

3.15. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$ -module.*

- (a) *Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in ${}^\circ\mathbf{R}_\nu$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.*
- (b) *There is a decomposition of graded ${}^\circ\mathbf{R}_\nu$ -modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.*
- (c) *We have a canonical isomorphism of ${}^\circ\mathbf{R}_\nu$ -modules $\text{res} \circ \text{ind}(M) = M \oplus M^\gamma$.*
- (d) *If there exists $a \in \{+, -\}$ such that $1_{\nu,-a}M = 0$, then there are canonical isomorphisms of graded ${}^\circ\mathbf{R}_\nu$ -modules*

$$M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma.$$

Proof: Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1({}^\theta I_+^\nu) = {}^\theta I_-^\nu$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b). \square

Now, let m and ν be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group W_m acts on ${}^\theta I^{\nu'}$, and the map

$${}^\theta I^\nu \rightarrow {}^\theta I^{\nu'}, \quad \mathbf{i} \mapsto \theta(i)\mathbf{i}\mathbf{i}$$

is W_m -equivariant. Thus there is $a_i \in \{+, -\}$ such that the image of ${}^\theta I_+^\nu$ is contained in ${}^\theta I_{a_i}^{\nu'}$, and the image of ${}^\theta I_-^\nu$ is contained in ${}^\theta I_{-a_i}^{\nu'}$.

3.16. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$ -module such that $1_{\nu,-a}M = 0$, with $a = +, -$. Then we have*

$$1_{\nu',-a_i a} f_i(M) = 0, \quad 1_{\nu',a_i a} f_{\theta(i)}(M) = 0.$$

Proof: We have

$$\begin{aligned} 1_{\nu',-a_i a} f_i(M) &= 1_{\nu',-a_i a} {}^\circ\mathbf{R}_{\nu'} 1_{\nu,i} \otimes_{{}^\circ\mathbf{R}_\nu} M \\ &= {}^\circ\mathbf{R}_{\nu'} 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} \otimes_{{}^\circ\mathbf{R}_\nu} M. \end{aligned}$$

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1_i)$ via the inclusion (3.2). The definition of this inclusion and that of a_i yield that

$$1_{\nu',a_i a} 1_{\nu,i} 1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} = 0.$$

The first equality follows. Next, note that for any $\mathbf{i} \in {}^\theta I^\nu$, the sequences $\theta(i)\mathbf{i}\mathbf{i}$ and $\mathbf{i}\mathbf{i}\theta(i) = \varepsilon_{m+1}(\theta(i)\mathbf{i}\mathbf{i})$ always belong to different ${}^\circ W_{m+1}$ -orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first. \square

Consider the following diagram

$$\begin{array}{ccc} {}^\circ\mathbf{R}_\nu\text{-mod} \times \mathbf{R}_i\text{-mod} & \begin{array}{c} \xrightarrow{\psi_!} \\ \xleftarrow{\psi^*} \end{array} & {}^\circ\mathbf{R}_{\nu'}\text{-mod} \\ \begin{array}{c} \uparrow \text{res} \times \text{id} \\ \downarrow \text{id} \times \text{id} \end{array} & & \begin{array}{c} \uparrow \text{res} \\ \downarrow \text{id} \end{array} \\ {}^\theta\mathbf{R}_\nu\text{-mod} \times \mathbf{R}_i\text{-mod} & \begin{array}{c} \xrightarrow{\psi_!} \\ \xleftarrow{\psi^*} \end{array} & {}^\theta\mathbf{R}_{\nu'}\text{-mod}. \end{array}$$

3.17. Lemma. *There are canonical isomorphisms of functors*

$$\begin{aligned} \text{ind} \circ \psi_! &= \psi_! \circ (\text{ind} \times \text{id}), & \psi^* \circ \text{ind} &= (\text{ind} \times \text{id}) \circ \psi^*, & \text{ind} \circ \psi_* &= \psi_* \circ (\text{ind} \times \text{id}), \\ \text{res} \circ \psi_! &= \psi_! \circ (\text{res} \times \text{id}), & \psi^* \circ \text{res} &= (\text{res} \times \text{id}) \circ \psi^*, & \text{res} \circ \psi_* &= \psi_* \circ (\text{res} \times \text{id}). \end{aligned}$$

Proof : The functor ind is left and right adjoint to res . Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

$$\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id})$$

comes from the associativity of the induction. Let us prove that

$$\psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*.$$

For any M in ${}^\circ\mathbf{R}_{\nu'}\text{-mod}$, the obvious inclusion ${}^\theta\mathbf{R}_\nu \otimes \mathbf{R}_i \subset {}^\theta\mathbf{R}_{\nu'}$ yields a map

$$(\text{ind} \times \text{id}) \psi^*(M) = ({}^\theta\mathbf{R}_\nu \otimes \mathbf{R}_i) \otimes_{{}^\circ\mathbf{R}_\nu \otimes \mathbf{R}_i} \psi^*(M) \rightarrow \psi^*({}^\theta\mathbf{R}_{\nu'} \otimes_{{}^\circ\mathbf{R}_\nu \otimes \mathbf{R}_i} M).$$

Combining it with the obvious map

$${}^\theta\mathbf{R}_{\nu'} \otimes_{{}^\circ\mathbf{R}_\nu \otimes \mathbf{R}_i} M \rightarrow {}^\theta\mathbf{R}_{\nu'} \otimes_{{}^\circ\mathbf{R}_{\nu'}} M$$

we get a morphism of ${}^\theta\mathbf{R}_\nu \otimes \mathbf{R}_i$ -modules

$$(\text{ind} \times \text{id}) \psi^*(M) \rightarrow \psi^* \text{ind}(M).$$

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

$$M \oplus (\pi_{1,\nu} \otimes M) \rightarrow M \oplus (\pi_{1,\nu'} \otimes M), \quad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu'} \otimes n.$$

Here $\pi_{1,\nu}$ and $\pi_{1,\nu'}$ denote the element π_1 in ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_{\nu'}$ respectively. □

3.18. Corollary. (a) *The operators e_i, f_i on ${}^\circ\mathbf{K}_{I,*}$ and on ${}^\theta\mathbf{K}_{I,*}$ are intertwined by the maps ind, res , i.e., we have*

$$e_i \circ \text{ind} = \text{ind} \circ e_i, \quad f_i \circ \text{ind} = \text{ind} \circ f_i, \quad e_i \circ \text{res} = \text{res} \circ e_i, \quad f_i \circ \text{res} = \text{res} \circ f_i.$$

(b) *The same result holds for the operators e_i, f_i on ${}^\circ\mathbf{G}_{I,*}$ and on ${}^\theta\mathbf{G}_{I,*}$.*

3.19. Now, we concentrate on non graded irreducible modules. First, let

$$\text{Res} : {}^\theta\mathbf{R}_\nu\text{-Mod} \rightarrow {}^\circ\mathbf{R}_\nu\text{-Mod}, \quad \text{Ind} : {}^\circ\mathbf{R}_\nu\text{-Mod} \rightarrow {}^\theta\mathbf{R}_\nu\text{-Mod}$$

be the (non graded) restriction and induction functors. We have

$$\mathbf{for} \circ \text{res} = \text{Res} \circ \mathbf{for}, \quad \mathbf{for} \circ \text{ind} = \text{Ind} \circ \mathbf{for}.$$

3.20. Lemma. *Let L, L' be irreducible ${}^\circ\mathbf{R}_\nu$ -modules.*

(a) *The ${}^\circ\mathbf{R}_\nu$ -modules L and L^γ are not isomorphic.*

(b) *$\text{Ind}(L)$ is an irreducible ${}^\theta\mathbf{R}_\nu$ -module, and every irreducible ${}^\theta\mathbf{R}_\nu$ -module is obtained in this way.*

(c) *$\text{Ind}(L) \simeq \text{Ind}(L')$ iff $L' \simeq L$ or L^γ .*

Proof: For any ${}^\theta\mathbf{R}_\nu$ -module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. The automorphism $M \rightarrow M$, $m \mapsto \pi_1 m$ takes $1_{\nu,+}M$ to $1_{\nu,-}M$ by Lemma 3.15(a). Hence $1_{\nu,-}M \neq 0$.

Now, to prove part (a), suppose that $\phi : L \rightarrow L^\gamma$ is an isomorphism of ${}^\circ\mathbf{R}_\nu$ -modules. We can regard ϕ as a γ -antilinear map $L \rightarrow L$. Since L is irreducible, by Schur's lemma we may assume that $\phi^2 = \text{id}_L$. Then L admits a ${}^\theta\mathbf{R}_\nu$ -module structure such that the ${}^\circ\mathbf{R}_\nu$ -action is as before and π_1 acts as ϕ . Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that L is an irreducible ${}^\circ\mathbf{R}_\nu$ -module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix]. \square

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in {}^\circ B$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module ${}^\theta b = \text{Ind}(b)$ lies in ${}^\theta B$. By [VV, prop. 8.2] there exists a unique selfdual irreducible graded ${}^\theta\mathbf{R}$ -module ${}^\theta G^{\text{up}}({}^\theta b)$ which is isomorphic to ${}^\theta b$ as a non graded module. Set

$${}^\circ G^{\text{up}}(b) = 1_{\nu,+} \text{res}({}^\theta G^{\text{up}}({}^\theta b)).$$

By Lemma 3.15(d) we have ${}^\circ G^{\text{up}}(b) = b$ as a non graded ${}^\circ\mathbf{R}$ -module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that M is another module with the same properties. Then $\text{ind}(M)$ is a selfdual graded ${}^\theta\mathbf{R}$ -module which is isomorphic to ${}^\theta b$ as a non graded ${}^\theta\mathbf{R}$ -module. Thus we have $\text{ind}(M) = {}^\theta G^{\text{up}}({}^\theta b)$ by loc. cit. By Lemma 3.15(d) we have also

$$M = 1_{\nu,+} \text{res}({}^\theta G^{\text{up}}({}^\theta b)).$$

So M is isomorphic to ${}^\circ G^{\text{up}}(b)$. \square

3.22. The crystal operators on ${}^\circ\mathbf{G}_I$ and ${}^\circ B$. Fix a vertex i in I . For each irreducible graded ${}^\circ\mathbf{R}_m$ -module M we define

$$\tilde{e}_i(M) = \text{soc}(e_i(M)), \quad \tilde{f}_i(M) = \text{top } \psi_i(M, \mathbf{L}_i), \quad \varepsilon_i(M) = \max\{n \geq 0; e_i^n(M) \neq 0\}.$$

3.23. Lemma. *Let M be an irreducible graded ${}^\circ\mathbf{R}$ -module such that $e_i(M), f_i(M)$ belong to ${}^\circ\mathbf{G}_{I,*}$. We have*

$$\text{ind}(\tilde{e}_i(M)) = \tilde{e}_i(\text{ind}(M)), \quad \text{ind}(\tilde{f}_i(M)) = \tilde{f}_i(\text{ind}(M)), \quad \varepsilon_i(M) = \varepsilon_i(\text{ind}(M)).$$

In particular, $\tilde{e}_i(M)$ is irreducible or zero and $\tilde{f}_i(M)$ is irreducible.

Proof: By Corollary 3.18 we have $\text{ind}(e_i(M)) = e_i(\text{ind}(M))$. Thus, since ind is an exact functor we have $\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M))$. Since ind is an additive functor, by Lemma 3.20(b) we have indeed

$$\text{ind}(\tilde{e}_i(M)) \subset \tilde{e}_i(\text{ind}(M)).$$

Note that $\text{ind}(M)$ is irreducible by Lemma 3.20(b), thus $\tilde{e}_i(\text{ind}(M))$ is irreducible by [VV, prop. 8.21]. Since $\text{ind}(\tilde{e}_i(M))$ is nonzero, the inclusion is an isomorphism. The fact that $\text{ind}(\tilde{e}_i(M))$ is irreducible implies in particular that $\tilde{e}_i(M)$ is simple. The proof of the second isomorphism is similar. The third equality is obvious. \square

Similarly, for each irreducible ${}^\circ\mathbf{R}$ -module b in ${}^\circ B$ we define

$$\tilde{E}_i(b) = \text{soc}(E_i(b)), \quad \tilde{F}_i(b) = \text{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geq 0; E_i^n(b) \neq 0\}.$$

Hence we have

$$\mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}, \quad \mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}, \quad \varepsilon_i = \varepsilon_i \circ \mathbf{for}.$$

3.24. Proposition. *For each b, b' in ${}^\circ B$ we have*

- (a) $\tilde{F}_i(b) \in {}^\circ B$,
- (b) $\tilde{E}_i(b) \in {}^\circ B \cup \{0\}$,
- (c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,
- (d) $\varepsilon_i(b) = \max\{n \geq 0; \tilde{E}_i^n(b) \neq 0\}$,
- (e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,
- (f) if $\tilde{E}_i(b) = 0$ for all i then $b = \phi_\pm$.

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23. \square

3.25. Remark. For any $b \in {}^\circ B$ and any $i \neq j$ we have $\tilde{F}_i(b) \neq \tilde{F}_j(b)$. This is obvious if $j \neq \theta(i)$. Because in this case $\tilde{F}_i(b)$ and $\tilde{F}_j(b)$ are ${}^\circ\mathbf{R}_\nu$ -modules for different ν . Now, consider the case $j = \theta(i)$. We may suppose that $\tilde{F}_i(b) = 1_{\nu,+}\tilde{F}_i(b)$ for certain ν . Then by Lemma 3.16 we have $1_{\nu,+}\tilde{F}_{\theta(i)}(b) = 0$. In particular $\tilde{F}_i(b)$ is not isomorphic to $\tilde{F}_{\theta(i)}(b)$.

3.26. The algebra ${}^\theta\mathbf{B}$ and the ${}^\theta\mathbf{B}$ -module ${}^\circ\mathbf{V}$. Following [EK1,2,3] we define a \mathcal{K} -algebra ${}^\theta\mathbf{B}$ as follows.

3.27. Definition. *Let ${}^\theta\mathbf{B}$ be the \mathcal{K} -algebra generated by e_i, f_i and invertible elements $t_i, i \in I$, satisfying the following defining relations*

- (a) $t_i t_j = t_j t_i$ and $t_{\theta(i)} = t_i$ for all i, j ,
- (b) $t_i e_j t_i^{-1} = v^{i \cdot j + \theta(i) \cdot j} e_j$ and $t_i f_j t_i^{-1} = v^{-i \cdot j - \theta(i) \cdot j} f_j$ for all i, j ,
- (c) $e_i f_j = v^{-i \cdot j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ for all i, j ,

$$(d) \quad \sum_{a+b=1-i \cdot j} (-1)^a e_i^{(a)} e_j e_i^{(b)} = \sum_{a+b=1-i \cdot j} (-1)^a f_i^{(a)} f_j f_i^{(b)} = 0 \text{ if } i \neq j.$$

Here and below we use the following notation

$$\theta^{(a)} = \theta^a / \langle a \rangle!, \quad \langle a \rangle = \sum_{l=1}^a v^{a+1-2l}, \quad \langle a \rangle! = \prod_{l=1}^m \langle l \rangle.$$

We can now construct a representation of ${}^\theta \mathbf{B}$ as follows. By base change, the operators e_i, f_i in Definition 3.8 yield \mathcal{K} -linear operators on the \mathcal{K} -vector space

$${}^\circ \mathbf{V} = \mathcal{K} \otimes_{\mathcal{A}} {}^\circ \mathbf{K}_I.$$

We equip ${}^\circ \mathbf{V}$ with the \mathcal{K} -bilinear form given by

$$(M : N)_{KE} = (1 - v^2)^m (M : N), \quad \forall M, N \in {}^\circ \mathbf{R}_m\text{-proj}.$$

3.28. Theorem. (a) The operators e_i, f_i define a representation of ${}^\theta \mathbf{B}$ on ${}^\circ \mathbf{V}$. The ${}^\theta \mathbf{B}$ -module ${}^\circ \mathbf{V}$ is generated by linearly independent vectors ϕ_+ and ϕ_- such that for each $i \in I$ we have

$$e_i \phi_{\pm} = 0, \quad t_i \phi_{\pm} = \phi_{\mp}, \quad \{x \in {}^\circ \mathbf{V}; e_j x = 0, \forall j\} = \mathbf{k} \phi_+ \oplus \mathbf{k} \phi_-.$$

(b) The symmetric bilinear form on ${}^\circ \mathbf{V}$ is non-degenerate. We have $(\phi_a : \phi_{a'})_{KE} = \delta_{a,a'}$ for $a, a' = +, -$, and $(e_i x : y) = (x : f_i y)_{KE}$ for $i \in I$ and $x, y \in {}^\circ \mathbf{V}$.

Proof: For each i in I we define the \mathcal{A} -linear operator t_i on ${}^\circ \mathbf{K}_I$ by setting

$$t_i \phi_{\pm} = \phi_{\mp} \quad \text{and} \quad t_i P = v^{-\nu \cdot (i + \theta(i))} P^\gamma, \quad \forall P \in {}^\circ \mathbf{R}_\nu\text{-proj}.$$

We must prove that the operators e_i, f_i , and t_i satisfy the relations of ${}^\theta \mathbf{B}$. The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey's induction-restriction theorem. Note that for $m > 1$ we have

$$D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1} \varepsilon_1\},$$

$$W(e) = {}^\circ W_m, \quad W(s_m) = {}^\circ W_{m-1}, \quad W(\varepsilon_{m+1} \varepsilon_1) = {}^\circ W_m.$$

Recall also that for $m = 1$ we have set ${}^\circ W_1 = \{e\}$.

3.29. Lemma. Fix i, j in I . Let μ, ν in ${}^\theta \mathbb{N}I$ be such that $\nu + i + \theta(i) = \mu + j + \theta(j)$. Put $|\nu| = |\mu| = 2m$. The graded $({}^\circ \mathbf{R}_{m,1}, {}^\circ \mathbf{R}_{m,1})$ -bimodule $1_{\nu,i} {}^\circ \mathbf{R}_{m+1} 1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to

$$\delta_{i,j}({}^\circ \mathbf{R}_\nu \otimes \mathbf{R}_i) \oplus \delta_{\theta(i),j}(({}^\circ \mathbf{R}_\nu)^\gamma \otimes \mathbf{R}_{\theta(i)})[d'] \oplus A[d],$$

where A is equal to

$$\begin{aligned} &({}^\circ \mathbf{R}_m 1_{\nu',i} \otimes \mathbf{R}_i) \otimes_{\mathbf{R}'} (1_{\nu',i} {}^\circ \mathbf{R}_m \otimes \mathbf{R}_i) && \text{if } m > 1, \\ &({}^\circ \mathbf{R}_{\theta(j)} \otimes \mathbf{R}_i \otimes {}^\circ \mathbf{R}_i \otimes \mathbf{R}_1 {}^\circ \mathbf{R}_{\theta(i)} \otimes \mathbf{R}_j) \oplus ({}^\circ \mathbf{R}_j \otimes \mathbf{R}_i \otimes {}^\circ \mathbf{R}_1 \otimes \mathbf{R}_1 {}^\circ \mathbf{R}_i \otimes \mathbf{R}_j) && \text{if } m = 1. \end{aligned}$$

Here we have set $\nu' = \nu - j - \theta(j)$, $\mathbf{R}' = {}^\circ\mathbf{R}_{m-1,1} \otimes \mathbf{R}_1$, $\mathbf{i} = i\theta(i)$, $\mathbf{j} = j\theta(j)$, $d = -i \cdot j$, and $d' = -\nu \cdot (i + \theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have

$$f_j(P) = {}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1), \quad e'_i(P) = 1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\circ\mathbf{R}_{m-1}$ -module. Therefore we have

$$\begin{aligned} e'_i f_j(P) &= 1_{m,i} {}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1), \\ f_j e'_i(P) &= {}^\circ\mathbf{R}_m 1_{m-1,j} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} (1_{m-1,i}P \otimes \mathbf{R}_1). \end{aligned}$$

Therefore, up to some filtration we have the following identities

- $e'_i f_i(P) = P \otimes \mathbf{R}_i + f_i e'_i(P)[-2]$,
- $e'_i f_{\theta(i)}(P) = P^\gamma \otimes \mathbf{R}_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_i(P)[-i \cdot \theta(i)]$,
- $e'_i f_j(P) = f_j e'_i(P)[-i \cdot j]$ if $i \neq j, \theta(j)$.

These identities also hold for $m = 1$ and $P = {}^\circ\mathbf{R}_{\theta(i)i}$ for any $i \in I$. The first claim of part (a) follows because $\mathbf{R}_i = \mathbf{k} \oplus \mathbf{R}_i[2]$. The fact that ${}^\circ\mathbf{V}$ is generated by ϕ_\pm is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b). \square

3.30. Remarks. (a) The ${}^\theta\mathbf{B}$ -module ${}^\circ\mathbf{V}$ is the same as the ${}^\theta\mathbf{B}$ -module V_θ from [KM, prop. 2.2]. The involution $\sigma : {}^\circ\mathbf{V} \rightarrow {}^\circ\mathbf{V}$ in [KM, rem. 2.5(ii)] is given by $\sigma(P) = P^\gamma$. It yields an involution of ${}^\circ B$ in the obvious way. Note that Lemma 3.20(a) yields $\sigma(b) \neq b$ for any $b \in {}^\circ B$.

(b) Let ${}^\theta\mathbf{V}$ be the ${}^\theta\mathbf{B}$ -module $\mathcal{K} \otimes_{\mathcal{A}} {}^\theta\mathbf{K}_I$ and let ϕ be the class of the trivial ${}^\theta\mathbf{R}_0$ -module \mathbf{k} , see [VV, thm. 8.30]. We have an inclusion of ${}^\theta\mathbf{B}$ -modules

$${}^\theta\mathbf{V} \rightarrow {}^\circ\mathbf{V}, \quad \phi \mapsto \phi_+ \oplus \phi_-, \quad P \mapsto \text{res}(P).$$

3.31. Proposition. *For any $b \in {}^\circ B$ the following holds*

$$(a) \quad \begin{cases} f_i({}^\circ G^{\text{low}}(b)) = \langle \varepsilon_i(b) + 1 \rangle {}^\circ G^{\text{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'} {}^\circ G^{\text{low}}(b'), \\ b' \in {}^\circ B, \quad \varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{2-\varepsilon_i(b')} \mathbb{Z}[v], \end{cases}$$

$$(b) \quad \begin{cases} e_i({}^\circ G^{\text{low}}(b)) = v^{1-\varepsilon_i(b)} {}^\circ G^{\text{low}}(\tilde{E}_i b) + \sum_{b'} e_{b,b'} {}^\circ G^{\text{low}}(b'), \\ b' \in {}^\circ B, \quad \varepsilon_i(b') \geq \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')} \mathbb{Z}[v]. \end{cases}$$

Proof: We prove part (a), the proof for (b) is similar. If ${}^\circ G^{\text{low}}(b) = \phi_\pm$ this is obvious. So we assume that ${}^\circ G^{\text{low}}(b)$ is a ${}^\circ\mathbf{R}_m$ -module for $m \geq 1$. Fix $\nu \in {}^\theta NI$

such that $f_i({}^\circ G^{\text{low}}(b))$ is a ${}^\circ \mathbf{R}_\nu$ -module. We'll abbreviate $1_{\nu,a} = 1_a$ for $a \in \{+, -\}$. Since ${}^\circ G^{\text{low}}(b)$ is indecomposable, it fulfills the condition of Lemma 3.16. So there exists $a \in \{+, -\}$ such that $1_{-a} f_i({}^\circ G^{\text{low}}(b)) = 0$. Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have

$$f_i({}^\circ G^{\text{low}}(b)) = 1_a \text{res ind } f_i({}^\circ G^{\text{low}}(b)) = 1_a \text{res } f_i \text{ ind } ({}^\circ G^{\text{low}}(b)).$$

Note that ${}^\theta b = \text{Ind}(b)$ belongs to ${}^\theta B$ by Lemma 3.20(b). Hence (3.5) yields

$$\text{ind}({}^\circ G^{\text{low}}(b)) = {}^\theta G^{\text{low}}({}^\theta b).$$

We deduce that

$$f_i({}^\circ G^{\text{low}}(b)) = 1_a \text{res } f_i({}^\theta G^{\text{low}}({}^\theta b)).$$

Now, write

$$f_i({}^\theta G^{\text{low}}({}^\theta b)) = \sum f_{\theta b, \theta b'} {}^\theta G^{\text{low}}({}^\theta b'), \quad \theta b' \in {}^\theta B.$$

Then we have

$$f_i({}^\circ G^{\text{low}}(b)) = \sum f_{\theta b, \theta b'} 1_a \text{res}({}^\theta G^{\text{low}}({}^\theta b')).$$

For any $\theta b' \in {}^\theta B$ the ${}^\circ \mathbf{R}$ -module $1_a \text{Res}({}^\theta b')$ belongs to ${}^\circ B$. Thus we have

$$1_a \text{res}({}^\theta G^{\text{low}}({}^\theta b')) = {}^\circ G^{\text{low}}(1_a \text{Res}({}^\theta b')).$$

If $\theta b' \neq \theta b''$ then $1_a \text{Res}({}^\theta b') \neq 1_a \text{Res}({}^\theta b'')$, because $\theta b' = \text{Ind}(1_a \text{Res}({}^\theta b'))$. Thus

$$f_i({}^\circ G^{\text{low}}(b)) = \sum f_{\theta b, \theta b'} {}^\circ G^{\text{low}}(1_a \text{Res}({}^\theta b')),$$

and this is the expansion of the lhs in the lower global basis. Finally, we have

$$\varepsilon_i(1_a \text{Res}({}^\theta b')) = \varepsilon_i({}^\theta b')$$

by Lemma 3.23. So part (a) follows from [VV, prop. 10.11(b), 10.16]. \square

3.32. The global bases of ${}^\circ \mathbf{V}$. Since the operators e_i, f_i on ${}^\circ \mathbf{V}$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$ on the ${}^\theta \mathbf{B}$ -module ${}^\circ \mathbf{V}$ as follows. For each u in ${}^\circ \mathbf{V}$ we write

$$u = \sum_{n \geq 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,$$

$$\tilde{\mathbf{e}}_i(u) = \sum_{n \geq 1} f_i^{(n-1)} u_n, \quad \tilde{\mathbf{f}}_i(u) = \sum_{n \geq 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let ${}^\circ \mathbf{L}$ be the \mathcal{R} -submodule of ${}^\circ \mathbf{V}$ spanned by the elements $\mathbf{f}_{i_1} \dots \mathbf{f}_{i_l}(\phi_\pm)$ with $l \geq 0, i_1, \dots, i_l \in I$. The following is the main result of the paper.

3.33. Theorem. (a) We have

$${}^\circ\mathbf{L} = \bigoplus_{b \in {}^\circ B} \mathcal{R} {}^\circ G^{\text{low}}(b), \quad \tilde{\mathbf{e}}_i({}^\circ\mathbf{L}) \subset {}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i({}^\circ\mathbf{L}) \subset {}^\circ\mathbf{L},$$

$$\tilde{\mathbf{e}}_i({}^\circ G^{\text{low}}(b)) = {}^\circ G^{\text{low}}(\tilde{E}_i(b)) \bmod v {}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i({}^\circ G^{\text{low}}(b)) = {}^\circ G^{\text{low}}(\tilde{F}_i(b)) \bmod v {}^\circ\mathbf{L}.$$

(b) The assignment $b \mapsto {}^\circ G^{\text{low}}(b) \bmod v {}^\circ\mathbf{L}$ yields a bijection from ${}^\circ B$ to the subset of ${}^\circ\mathbf{L}/v {}^\circ\mathbf{L}$ consisting of the $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$'s. Further ${}^\circ G^{\text{low}}(b)$ is the unique element $x \in {}^\circ\mathbf{V}$ such that $x^\sharp = x$ and $x = {}^\circ G^{\text{low}}(b) \bmod v {}^\circ\mathbf{L}$.

(c) For each b, b' in ${}^\circ B$ let $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ be the coefficients of ${}^\circ G^{\text{low}}(b')$ in $e_{\theta(i)}({}^\circ G^{\text{low}}(b))$, $f_i({}^\circ G^{\text{low}}(b))$ respectively. Then we have

$$E_{i,b,b'}|_{v=1} = [F_i \Psi \mathbf{for}({}^\circ G^{\text{up}}(b')) : \Psi \mathbf{for}({}^\circ G^{\text{up}}(b))],$$

$$F_{i,b,b'}|_{v=1} = [E_i \Psi \mathbf{for}({}^\circ G^{\text{up}}(b')) : \Psi \mathbf{for}({}^\circ G^{\text{up}}(b))].$$

Proof: Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], and Proposition 3.31. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard $E_{i,b,b'}$, $F_{i,b,b'}$ as the coefficients of ${}^\circ G^{\text{up}}(b)$ in $f_{\theta(i)}({}^\circ G^{\text{up}}(b'))$ and $e_i({}^\circ G^{\text{up}}(b'))$ respectively. Therefore, by Proposition 3.11 we can regard $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi \mathbf{for}({}^\circ G^{\text{up}}(b))$ in $F_i \Psi \mathbf{for}({}^\circ G^{\text{up}}(b'))$ and $E_i \Psi \mathbf{for}({}^\circ G^{\text{up}}(b'))$ respectively. \square

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